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By *Walter Briec and Laurence Lasselle*

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WP Nº 14-011

3rd Quarter 2014

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11 August 2014

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We introduce well-known microeconomics productivity measures in Solow models in discrete time or continuous time by adopting a Luenberger-type approach. In each framework, we derive the productivity indicators and the dynamical paths. Firstly, we show that the expression of the productivity indicators are similar to the well-known Solow residuals, allowing us to make an analogy between a firm's behaviour in a microeconomic setting and a country's behaviour in a macroeconomic setting. Secondly, we demonstrate that the properties of the paths are similar in both frameworks. Finally, we develop a new class of distance functions, the exponential distance, which facilitates the productivity analysis in these models.

JEL: D24
Keywords: technical efficiency, technological progress, exponential distance, proportional distance.

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1 Introduction

1957 is a landmark for the understanding of productivity. On the one hand, Solow (1957) proposed the first macroeconomic framework to evaluate the total factor productivity indices from a neoclassical aggregate production function. Solow's research initiated modern growth accounting research that will lead to distinguish 'separate contributions of technological change, capital accummulation, education and so forth in raising per capita income' (Allen, 1991, p 203). It is now well-established that productivity growth has two determinants: variation of technology efficiency (the so-called 'productivity gains') and technological changes linked to innovation. On the other hand, Farrell (1957) proposed various measurements of productive efficiency at the firm level by using the concept of distance function. A firm is said to be technically inefficient if it can produce the same output with an equiproportional reduction in the use of all inputs. Procedures to measure inefficiency have since flourished in the literature.

The aim of this paper is to unify both approaches in simple macrodynamical frameworks. Our paper shows the analogy between technical efficiencies relative to production sets and the 'standard' theory of productivity measurement derived from Solow. Indeed, technical efficiency in Farrell's sense is the total factor productivity by a different name. Solow's productivity growth is the total factor productivity with a shift in technology. This correspondence is not new. Indeed, as noted by ten Raa and Mohnen (2002, p. 111), 'productivity is essentially the output-input ratop and therefore productivity growth the residual between output growth and input growth' in both approaches. Introducing the distance function concept in macrodynamics has two main advantages. On the one hand, estimating total factor productivity growth does not require the specification of the production function. On the other hand, evaluating production inefficiencies and therefore identifying possible aggregate gains become possible.

To illustrate the above, let us assume that a firm is efficient at a given time period. If the set of production possibilities is increasing in time, then it may not remain efficient relative to the technical efficiency frontier at a later time period. As it is in the firm's interest to maintain efficiency, the firm has to change size between these two time periods. It does so by proceeding to a minimal transformation of factors and products. This 'ideal' change between two successive time periods can be evaluated by using the procedures derived from Farrell. The firm's dynamical behavior can then be extrapolated by recurrence, although it remains dependent on its initial condition. We can make use of similar arguments in the case a country in a macrodynamical framework.

In this paper, we make the analogy between the Farrell and Solow productivity concepts in simple Solow models. This allows us to propose an alternative way to measure the performance of decision units, including countries, for a given exogenous technological progress. The basic correspondence is presented by Del Gatto, Di Liberto and Petraglia (2011, pp. 962-974). Our paper does not explain the origin of the technological progress. Our (macroeconomic) models do not have microeconomic foundations. Specifically, we consider simple Solow-type models in which the technological progress is exogenous and the technical efficiency frontier is not specified. The dynamical analysis evaluates the impact of a firm's size change on productivity over a period of time, by simply actualizing observations over time. This enables us to treat these observations as cross-sectional data to estimate the '*actualized*' frontier by envelopment. Although our concept of optimal paths is close to that derived in the traditional macroeconomic approach, it rests on the distance concept developed by Luenberger (1995) and Chambers, Chung and Färe (1996 and 1998). In other words, we integrate some essential and well-known tools of production theory in macrodynamics.

The paper is organized as follows. Section 2 collates the basic assumptions and definitions. We determine the dynamical path according to the rate of growth of the technological progress in discrete time in Section 3 and in continuous time in Section 4. Section 5 proposes a new class of distance function, the exponential distance, which facilitates the productivity analysis in macrodynamics, the exponential distance function. Section 6 concludes.

2 Technology, technological progress and distance function

Firstly, this section introduces the assumptions on the production possibility set. Secondly, it presents the methods to evaluate efficiency changes relative to production frontiers, including the indicators used throughout the paper.

2.1 Assumptions on production technology

For $x, u \in \mathbb{R}^n_+$, $x \leq u \Leftrightarrow x_i \leq u_i, \forall\, i \in \{1, ..., n\}$. A production technology transforms input vectors $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$ into output vectors $y = (y_1, ..., y_p) \in \mathbb{R}^p_+$. This production technology can be characterized by the input correspondence $L : \mathbb{R}^p_+ \longrightarrow 2^{\mathbb{R}^n_+}$ or the output correspondence $P : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^p_+}$. For all $y \in \mathbb{R}^p_+$, $L(y)$ is the set of all input vectors that yield at least y. For all $x \in \mathbb{R}^n_+$, $P(x)$ is the set of all output vectors obtainable from a given input vector x. The graph of the technology T can be defined either from L or P by:

$$T = \big\{(x, y) \in \mathbb{R}^{n+p}_+ : x \in L(y)\big\} = \big\{(x, y) \in \mathbb{R}^{n+p}_+ : y \in P(x)\big\}.$$

As Färe, Grosskopf and Lovell (1985) and Shephard (1974), we assume that the technology representation satisfies the standard axioms of production:
T1: $(0, 0) \in T$; $(0, y) \in T \Rightarrow y = 0$.
T2: $T(x) = \{(u, y) \in T : u \leq x\}$ is a bounded set $\forall x \in \mathbb{R}^n_+$.
T3: $\forall\ (u, v) \in \mathbb{R}^{n+p}_+; (x, y) \in T \wedge (-u, v) \leq (-x, y) \Rightarrow (u, v) \in T$.

T4: T is a closed set.

Axiom T1 states that outputs cannot be produced without inputs. Although this axiom is realistic, it is not compulsory to maintain our results. Axiom T2 postulates that a finite input vector cannot lead to an infinite output vector. Strong disposability of both inputs and outputs is imposed by Axiom T3. Axiom T4 is a required condition to identify the set of efficient vectors of a subset of the frontier. To Axioms T1-T4 can be added Axiom T5 postulating the convexity of the production set, often used in empirical works (see for instance Charnes, Cooper and Rhodes (1978) and Banker, Charnes and Cooper (1984)).

T5: T is a convex set.

In what follows, we will assume that the technology T satisfies the axioms T1-T5, unless stated otherwise.

2.2 Technical efficiencies and distance functions

Any difference between actual production set and efficient set reveals efficient gains possibilities. In practice, these are evaluated by using distance functions that measure technical efficiencies relative to technology subsets. This paper considers the proportional distance function based on simultaneous proportionate changes in inputs and outputs. Initiated by Briec (1997), this distance function is closely related to the directional distance function due to Chambers, Chung and Färe (1996,1998).[1] This specification has the advantage to take into account various weightings on inputs and outputs. It can be justified in a macroeconomic framework as it could measure changes in per capita output relative to changes in per capita labor for instance.

Definition 2.2.1 *The map $D_T^\propto : \mathbb{R}_+^{n+p} \times [0,1]^{n+p} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by*

$$D_T^\propto(x, y; \alpha, \beta) = \max \quad \{\delta : (x - \delta\alpha \odot x, y + \delta\beta \odot y) \in T\} \tag{2.1}$$

is called Proportional Distance Function, where $\odot$ denotes the coordinate-wise product defined by $\gamma \odot z = (\gamma_1 z_1, ..., \gamma_d z_d)$ for all $\gamma, z \in \mathbb{R}^d$.

Notice that, for all $(\alpha, \beta) \in [0,1]^{n+p}$, this distance function is independent of the units of measurement chosen for the production technology. Equivalently, this means that it satisfies the commensurability condition (see Russell (1987)).

[1] Russell and Schworm (2011) mention some differences between these distance functions.

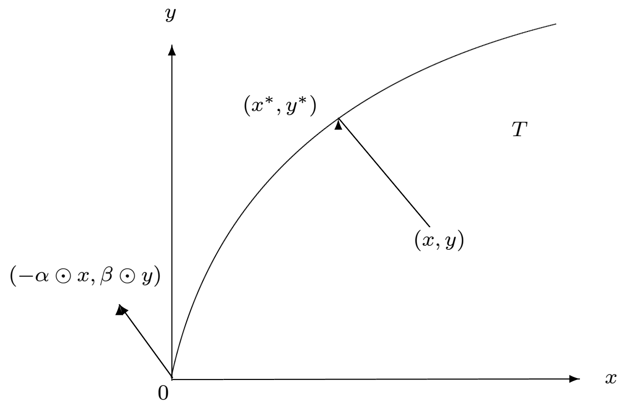

Figure 1: Proportional distance function.

Figure 1 illustrates the concept of proportional distance function in a two-dimension diagram where $\alpha \odot x = \alpha x$ and $\beta \odot y = \beta y$ with $\alpha > 0$ and $\beta > 0$. More precisely, the proportional distance $D_T^{\propto} : \mathbb{R}_+^2 \times [0,1]^2 \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $D_T^{\propto}(x, y; \alpha, \beta) = \max\{\delta : (x - \delta\alpha\ x, y + \delta\beta\ y) \in T\}$. The coordinates of the optimal projection point $(x^\star, y^\star)$ are $(x - \alpha\delta^\star x, y + \beta\delta^\star y)$ where $\delta^\star = D_T^{\propto}(x, y; \alpha, \beta)$, from which the following proportional equality is deduced

$$\frac{x - x^\star}{\alpha x} = \frac{y^\star - y}{\beta y}. \tag{2.2}$$

Equality 2.2 indicates the necessary proportional changes in input and output in order to make the input-output vector (x, y) efficient.

From Definition 2.2.1, we can retrieve the two well-known Farrell measures of technical efficiency. The Farrell input measure of technical efficiency $E^i : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$E^i(x, y) = \inf\{\lambda > 0 : \lambda x \in L(y)\} \tag{2.3}$$

The Farrell output measure of technical efficiency $E^o : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R}_+ \cup \{-\infty, +\infty\}$ is defined by

$$E^o(x, y) = \sup\{\theta > 0 : \theta y \in P(x)\}. \tag{2.4}$$

In addition, if we set $\alpha = \mathbb{1}^n$ and $\beta = 0$, we have $D_T^{\propto}(x, y; \mathbb{1}^n, 0) = 1 - E^i(x, y)$. If we set $\alpha = 0$ and $\beta = \mathbb{1}^n$, we have $D_T^{\propto}(x, y; 0, \mathbb{1}^n) = E^o(x, y) - 1$.

To measure technical efficiency in simple macrodynamical frameworks with exogenous growth, let us give an intertemporal dimension to our framework (*cf.* Definition 2.2.3) by from the subset of the frontier T (*cf.* Definition 2.2.2).

Definition 2.2.2 *The subset*

$$\partial^{\propto}_{\alpha,\beta}(T) = \big\{(x,y) \in T : \delta > 0 \Longrightarrow (x - \delta\alpha \odot x, y + \delta\beta \odot y) \notin T\big\} \tag{2.5}$$

is called the oriented Graph-Isoquant of the production set T.

From Definition 2.2.2, we deduce

$$D^{\propto}_{T}(x,y;\alpha,\beta) = 0 \Leftrightarrow (x,y) \in \partial^{\propto}_{\alpha,\beta}(T). \tag{2.6}$$

In other words, any input-output vector on the efficiency frontier, *i.e.* the oriented-graph isoquant, has a null distance.

Definition 2.2.3 *The set of production possibilities $T(t)$ for each time period t is defined by*

$$T(t) = \big\{(x(t), y(t)) \in \mathbb{R}^{n+p}_{+} : (x(t), y(t)) \textit{ is possible at time period } t\big\}.$$

In what follows, we will denote $D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big)$, the proportional distance function relative to the technology $T(t)$ at time period t and $\partial^{\propto}_{\alpha,\beta}(T(t))$, the Graph-Isoquant of $T(t)$ at time period t. The input-output vector $(x(t), y(t))$ can also be called the production unit. We will assume that $T(t)$ satisfies Axioms T1-T5 at each time period t, unless stated otherwise.

2.3 Technological progress and proportional Luenberger indicator

In this sub-section, we define the proportional Luenberger indicator,[2] our productivity indicator based on the proportional distance function. This indicator provides a flexible tool to account for both input contractions and output improvements when measuring efficiency (see Boussemart, Briec, Kerstens and Poutineau (2003, p. 393) for more information on Luenberger indexes and indicators).

In our framework, the set of production possibilities, defining the technological constraints supported by the producer, is assumed to increase in time for a given amount of input. In other words, if the producer adopts a given size in the market at a given time period, she will be able to do so again at a later time period. This is gathered in the following assumption.

Assumption (Growth of the set of production possibilities)

$$\forall\, t, s \geq 0 \quad \text{if} \quad s \geq t \quad \text{then} \quad T(t) \subset T(s).$$

[2] Our productivity indicator was first introduced by Luenberger (1992a and 1992b) and subsequently developed by Chambers, Färe and Grosskopf (1996).

The proportional Luenberger indicator is equal to

$$\begin{aligned}
L_{t,t+1}&\big(x(t),y(t),x(t+1),y(t+1);\alpha,\beta\big)\\
&=\frac{1}{2}\Big[\Big(D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big)-D^{\propto}_{T(t)}\big(x(t+1),y(t+1);\alpha,\beta\big)\Big)\\
&\quad+\Big(D^{\propto}_{T(t+1)}\big(x(t),y(t);\alpha,\beta\big)-D^{\propto}_{T(t+1)}\big(x(t+1),y(t+1);\alpha,\beta\big)\Big)\Big]
\end{aligned}\tag{2.7}$$

Indicator 2.7 is the arithmetic mean between the proportional changes in input and output observed at time period t and those observed at time period $t+1$. This allows us to avoid an arbitrary choice between base years. For instance, $D^{\propto}_{T(t+1)}\left(x(t),y(t);\alpha,\beta\right)$ indicates the necessary proportional changes in inputs and outputs for $(x(t),y(t))$ to be efficient at time period $t+1$. If $L=0$, there are no productivity gains. If $L>0$, there are productivity gains. If $L<0$, there are productivity losses. As this Luenberger indicator evaluates the productivity change, we denote it by PCH. The latter can be decomposed into two components: the efficiency change ($EFCH$) of the proportional distance function between the two successive time periods and the technological change ($TECH$) measured by the arithmetic mean of the last two differences. Hence, it can be expressed as:

$$PCH_{t,t+1}=EFCH_{t,t+1}+TECH_{t,t+1}\tag{2.8}$$

where

$$EFCH_{t,t+1}=D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big)-D^{\propto}_{T(t+1)}\big(x(t+1),y(t+1);\alpha,\beta\big)\tag{2.9}$$

and

$$\begin{aligned}
TECH_{t,t+1}&=\frac{1}{2}\Big[\Big(D^{\propto}_{T(t+1)}\big(x(t),y(t);\alpha,\beta\big)-D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big)\Big)\\
&\quad+\Big(D^{\propto}_{T(t+1)}\big(x(t+1),y(t+1);\alpha,\beta\big)-D^{\propto}_{T(t)}\big(x(t+1),y(t+1);\alpha,\beta\big)\Big]
\end{aligned}\tag{2.10}$$

Expression 2.10 does not necessitate the specification of the production function. As well-known in the literature, the latter can be estimated from non parametric techniques.

2.4 Example

Let us consider a Cobb-Douglas production function with constant returns to scale. The expression of the technology at time period t is given by

$$T(t)=\Big\{(x(t),y(t))\in\mathbb{R}^{n+1}_{+}:y(t)\leq A(t)\prod_{i=1}^{n}x_i(t)^{\gamma_i}\Big\}\text{ with }\gamma_n>0\text{ and }\sum_{i=1}^{n}\gamma_i=1.\tag{2.11}$$

We assume that all firms are efficient, *i.e.* $y(t) = A(t)\prod_{i=1}^{n} x_i(t)^{\gamma_i}$.

Let us first set $\alpha = \mathbb{1}^n$ and $\beta = 0$. As $D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) = 1 - \frac{A(t)\prod_{i=1}^{n} x_i(t)^{\gamma_i}}{y(t)}$ and $y(t) = A(t)\prod_{i=1}^{n} x_i(t)^{\gamma_i}$, it is easy to show that the proportional efficiency change is null, *i.e.* $EFCH_{t,t+1} = 0$. The technological change is equal to

$$TECH_{t,t+1} = \frac{1}{2}\Big[\frac{A(t+1) - A(t)}{A(t+1)} + \frac{A(t+1) - A(t)}{A(t)}\Big]. \tag{2.12}$$

As a result, $PCH_{t,t+1} = EFCH_{t,t+1} + TECH_{t,t+1} = TECH_{t,t+1}$. If $t \simeq t+1$, (2.12) becomes

$$TECH_{t,t+1} = \Big[\frac{A(t+1) - A(t)}{A(t+1)}\Big] \simeq \frac{\dot{A}(t)}{A(t)} \simeq \frac{d\log A(t)}{dt} = d\log A \tag{2.13}$$

Let us simplify the notation by denoting $\frac{d\log A(t)}{dt}$ by $d\log A$.

Recall that in macrodynamical frameworks, when the production function is specified, the technological change can be deduced from the Log-transformation of (2.11). It yields

$$\log\left(A(t)\right) = \log y(t) - \sum_{i=1}^{n} \gamma_i \log\left(x_i\left(t\right)\right) \tag{2.14}$$

and by taking the discrete time approximation of the technological progress as suggested by Solow, we obtain:

$$d\log\left(A\right) = d\log\left(y\right) - \sum_{i=1}^{n} \gamma_i d\log\left(x_i\right) \tag{2.15}$$

This is the so-called Solow residual. As $PCH_{t,t+1} = TECH_{t,t+1} = d\log A$, we can deduce that the Solow apprach and the Farrell approach are equivalent.

Let us now set $\alpha = \mathbb{1}^n$ and $\beta = \mathbb{1}^n$. The expression of the proportional distance function becomes

$$\begin{aligned}
D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) &= \max_{\delta}\Big\{\delta \geq 0; (1+\delta)\, y(t) \leq A(t)\prod_{i=1}^{n}\left((1-\delta)\, x_i(t)\right)^{\gamma_i}\Big\} && (2.16)\\
&= \max_{\delta}\Big\{\delta \geq 0; \frac{(1+\delta)}{(1-\delta)} y(t) \leq A(t)\prod_{i=1}^{n}\left(x_i(t)\right)^{\gamma_i}\Big\} && (2.17)\\
&= \frac{A(t)\prod_{i=1}^{n} x_i(t)^{\gamma_i} - y(t)}{A(t)\prod_{i=1}^{n} x_i(t)^{\gamma_i} + y(t)} && (2.18)
\end{aligned}$$

from which the expressions of $EFCH_{t,t+1}$, $TECH_{t,t+1}$ and $PCH_{t,t+1}$ can be calculated.

3 Dynamical path in discrete time

In this section, we assume that the size of an efficient firm at time period t evolves in such a way that it is still efficient at the successive time period. In what follows, the output vector at time period $t+1$ always 'realizes' its proportional distance of the previous time period t in our framework.[3] Without this assumption, the output vector relatively efficient to the technical efficiency frontier at time period t would not be automatically efficient at time period $t+1$. It is only a size change that allows the decision unit to be on the frontier at the following time period. If the output vector systematically follows this transformation rule between two successive time periods, then there exists an optimal dynamical behavior.

3.1 Notations and definitions

The technical efficiency indicator of the input-output $(x(t), y(t))$ at time period s is the distance $D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta)$. This indicator depends on $T(s)$, *i.e.* the production set at a given time period.

Let $\mathfrak{T} \subset \mathbb{R}_+$. The *family of production sets* $\{T(t)\}_{t\in\mathfrak{T}}$ is the set of all production sets $T(t)$ for $t \in \mathfrak{T}$. This definition is independent on the time characteristics. It can be used in discrete time ($t \in \mathbb{N}$) or in continuous time ($t \in \mathbb{R}_+$). The family $\{T(t)\}_{t\in\mathfrak{T}}$ is clearly embedded because of Axiom T1.

The *dynamical path* $\{(x(t), y(t))\}_{t\in\mathfrak{T}}$ of the production unit (x, y) is the set of the successive positions of the production unit (x, y) over time.

The optimal dynamical path of a production unit always depends on the choice of the technical efficiency indicator. In our context, the dynamical path of a vector is said to be 'optimal' if its outcome is an optimal movement between two successive time periods for a given proportional distance.

In what follows, we denote (u, v) the reference of (x, y) at time period t if

$$(u, v) = (x, y) + \left[D^{\propto}_{T(t)}(x, y; \alpha, \beta)\right](-\alpha \odot x, \beta \odot y). \tag{3.1}$$

Definition 3.1.1 *Let $\{T(t_k)\}_{k=0,\ldots,m}$ be a family of production sets ($t_0 < t_1 < t_2 < \ldots < t_m$) satisfying Axioms T1-T5. Let $\{x(t_k), y(t_k)\}_{k=0,\ldots,m}$ be a dynamical path of (x, y).*

If at each time period t_k, $(x(t_{k+1}), y(t_{k+1}))$ realizes the value of efficiency indicator of $(x(t_k), y(t_k))$ at time period t_{k+1} on the oriented-graph isoquant $\partial^{\propto}_{\alpha,\beta}(T(t_{k+1}))$, then the path $\{x(t_{k+1}), y(t_{k+1})\}_{k=0,\ldots,m}$ is optimal.

[3] This assumption is not original. A similar assumption can be found in Caves, Christensen and Diewert (1982), Färe, Grosskopf, Lindgren and Roos (1992), and Tulkens and Vanden Eeckaut (1995).

3.2 Time interval between two periods is regular

Definition 3.1.1 allows us to introduce the concept of regularity, commonly used in the macroeconomics literature on technological progress.

Definition 3.2.1 *An optimal path is θ-regular if there is $\theta > 0$ such that*

$$D^{\propto}_{T(t_{k+1})}(x(t_k), y(t_k); \alpha, \beta) = \theta$$

for all $k = 0, ..., m-1$.

The regularity specified in Definition 3.2.1 is similar to that considered in growth models. In these latter, regularity is associated with the dynamical paths characterising technological progress through a linear time-dependent exponential function. The production process is of course modeled over time and the productivity measures evaluate by how much output results from a particular level of inputs. This specification can be studied in terms of variation rate of the factors or that of the products. This is why we need the following lemma. For all $(x, y) \in \mathbb{R}^{n+p}_+$, let us denote $I(x) = \{i \in [n] : x_i > 0\}$ and $J(y) = \{j \in [p] : y_j > 0\}$.

Lemma 3.2.2 *Let $T(t)$ and $T(s)$ be two production possibility sets at time periods t and s with $s \geq t$, both satisfying Axioms T1-T4. Let $(x(t), y(t))$ and $(x(s), y(s))$ be respectively efficient production units at time periods t and s. We have for all $(i, j) \in I(x(t)) \times J(y(t))$,*

$$D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta) = \frac{1}{\alpha_i}\frac{x_i(t) - x_i(s)}{x_i(t)} = \frac{1}{\beta_j}\frac{y_j(s) - y_j(t)}{y_j(t)}.$$

Moreover,

(1) if $\alpha = \mathbb{1}^n$ and $\beta = 0$, then $D^{\propto}_{T(s)}(x(t), y(t); \mathbb{1}^n, 0)$ is equal to the growth rate of the products between two successive time periods;

(2) if $\alpha = 0$ and $\beta = \mathbb{1}^n$, then $D^{\propto}_{T(s)}(x(t), y(t); 0, \mathbb{1}^p)$ is equal to the decline rate of the factors between two successive time periods;

(3) if $\alpha = \mathbb{1}^n$ and $\beta = \mathbb{1}^n$, then $D^{\propto}_{T(s)}(x(t), y(t); \mathbb{1}^n, \mathbb{1}^p)$ is equal to the decline rate of the factors and the growth rate of the products between two successive time periods.

Proof: We know

$$\big(x(s), y(s)\big) = (x(t), y(t)) + \left[D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta)\right](-\alpha \odot x(t), \beta \odot y(t)).$$

To simplify the notation, let us denote $d(t, T(s))$ the expression of the proportional distance $D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta)$. The above expression can be rewritten as

$$x(s) = x(t) - d(t, T(s))\alpha_i x(t) \quad \text{and} \quad y(s) = y(t) + d(t, T(s))\beta_j y(t) \text{ for all } i \text{ and } j.$$

The expression of the distance functions then follows.

Let $\alpha = \mathbb{1}^n$, $\beta = \mathbb{1}^n$ and $(x(s), y(s)) = \big([1 - d(t, T(s))]x(t), [1 + d(t, T(s))]y(t)\big)$ be the reference of $(x(t), y(t))$ at time period s. We can easily deduce that for all $i \in I(x(t))$ and all $j \in J(y(t))$ $\frac{-x_i(s) + x_i(t)}{x_i(t)} = d(t, T(s))$ and $\frac{y_j(s) - y_j(t)}{y_j(t)} = d(t, T(s))$.□

3.3 Time interval between two periods is not regular

The possible irregularity of the time interval does not prevent us from constructing a recurrent process determining the dynamics of the production set. Let A and B be respectively the diagonal matrices of α and β.

Lemma 3.3.1 *Let $\{T(t_k)\}_{k=0,...,m}$ be a family of production sets ($t_0 < t_1 < t_2 < ... < t_m$) satisfying Axioms T1-T4. Let $\{(x(t_k), y(t_k)\}_{k=0,1,...,m}$ be the optimal dynamical path. If at each time period t_k, $(x(t_{k+1}), y(t_{k+1}))$ is the reference of $(x(t_k), y(t_k))$ relatively to t_{k+1}, then we have at time period t_m*

$$x(t_m) = \Big[\prod_{k=0}^{m-1}(I - \theta_k A)\Big]x(t_0) \text{ and } y(t_m) = \Big[\prod_{k=0}^{m-1}(I + \theta_k B)\Big]y(t_0)$$

where $\theta_k = D^{\propto}_{T(t_{k+1})}(x(t_k), y(t_k); \alpha, \beta)$ for all $k = 0, ..., m-1$.

Proof: By definition of the proportional distance, we have

$$x(t_m) = (\mathbb{1} - \theta_{m-1}\alpha) \odot x(t_{m-1}) \text{ and } y(t_m) = (\mathbb{1} + \theta_{m-1}\beta) \odot y(t_{m-1}).$$

Equivalently,

$$x(t_m) = (I - \theta_{m-1}A)x(t_{m-1}) \text{ and } y(t_m) = (I + \theta_{m-1}B)y(t_{m-1}).$$

We also have

$$x(t_{m-1}) = (I - \theta_{m-2}A)x(t_{m-2}) \text{ and } y(t_{m-1}) = (1 + \theta_{m-2}B)y(t_{m-2}).$$

By recurrence, we deduce the result.□

As the regular time interval is a special case of irregular time intervals, the following two corollaries can be deduced from Lemma 3.3.1.

Corollary 3.3.2 *Let $\{T(t)\}_{t\in\mathbb{N}}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t)\}_{t\in\mathbb{N}}$ be the optimal dynamical path. If $\{(x(t), y(t)\}_{t\in\mathbb{N}}$ is θ-regular, then we have at each time period t*

$$x(t) = (1 - \theta)^t x(0) \text{ and } y(t) = (1 + \theta)^t y(0)$$

where θ is the decline rate of the factors and the growth rate of product between two successive time periods.

Proof: From Lemma 3.3.1, if we consider the family of production sets $\{T(t)\}_{t\in\{t_0,t_1,....,t_m\}}$ and an identical θ we have

$$x(t_m) = \left[\prod_{k=0}^{m-1}(I - \theta I)^k\right] x(t_0) \text{ and } y(t_m) = \left[\prod_{k=0}^{m-1}(I + \theta I)^k\right] y(t_0).$$

Setting $t_0 = 0$, $t_1 = 1$, *etc.* yields

$$x(t) = (1-\theta)^t x(0) \text{ and } y(t) = (1+\theta)^t y(0). \square$$

The above expression reminds us the usual specifications in simple macrodynamical frameworks. The simplest Solow framework has only one product, the output per capita. The latter grows according to the technological progress, *i.e.* $y_t = (1+\theta)^t F(x_0)$ where F is the aggregate production function and θ the growth rate of the product at each time period. If we consider a slightly more sophisticated model in which we assume a decline rate of the factors and a growth rate of the product, the per capita output grows according to $y_t = (1+\theta)^t F((1-\sigma)^t x_0)$.

The second corollary identifies the correspondence between regularity and proportionality.

Corollary 3.3.3 *Let $\{T(t)\}_{t\in\mathbb{N}}$ be a family of production sets satisfying Axioms T1-T4. Let $\{x(t), y(t)\}_{t\in\mathbb{N}}$ be the optimal dynamical path. If $\{(x(t), y(t)\}_{t\in\mathbb{N}}$ is θ-regular, the two following properties apply.*

(1) *For each time period t, the factors are used in the same proportion.*

(2) *For each time period t, the products are produced in the same proportion.*

Proof:

(1) From Lemma 3.3.1, $x(t) = (1-\theta)^t x(0)$ at time period t. It then results $\forall i, j \in I(x(0))^2$,

$$\frac{x_i(t)}{x_j(t)} = \frac{(1-\theta)^t x_i(0)}{(1-\theta)^t x_j(0)} = \frac{x_i(0)}{x_j(0)}.$$

In other words, the proportion by which each factor is used is constant over time.

(2) The proof is similar. $\square$

In the case of non-regular path, *i.e.* $s \neq t$, one can extend the definition of the Luenberger indicator as:

$$\begin{aligned} &L_{t,s}\big(x(t), y(t), x(s), y(s); \alpha, \beta\big) \\ &\quad = \frac{1}{2(s-t)}\Big[\Big(D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)\Big) \\ &\qquad + \Big(D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)\Big)\Big] \end{aligned} \tag{3.2}$$

As in Section 2, we denote this proportional Luenberger indicator, $PCH_{t,s}$. The latter can be decomposed into two components: the proportional efficiency change, $EFCH_{t,s}$, and the proportional technological change, $TECH_{t,s}$. It can be expressed as

$$PCH_{t,s} = EFCH_{t,s} + TECH_{t,s} \tag{3.3}$$

where

$$EFCH_{t,s} = \frac{1}{(s-t)}\Big[D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(s)}\big(x(s),y(s);\alpha,\beta\big)\Big] \tag{3.4}$$

and

$$\begin{aligned} TECH_{t,s} = \frac{1}{2(s-t)}\Big[&\Big(D^{\propto}_{T(s)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big)\Big) \\ &+ \Big(D^{\propto}_{T(s)}\big(x(s),y(s);\alpha,\beta\big) - D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)\Big)\Big] \end{aligned} \tag{3.5}$$

Corollary 3.3.4 *Let $\{T(t_k)\}_{k=0,\dots,m}$ be a family of production sets ($t_0 < t_1 < t_2 < \dots < t_m$) satisfying Axioms T1-T4. Let $\{(x(t_k), y(t_k)\}_{k=0,1,\dots,m}$ be the optimal dynamical path. If at each time period t_k, $(x(t_{k+1}), y(t_{k+1}))$ is the reference of $(x(t_k), y(t_k))$ relatively to t_{k+1}, then we have at time period t_m*

$$L\big(x(t_k), y(t_k), x(t_{k+1}), y(t_{k+1}); \alpha, \beta\big) = \theta_k$$

where $\theta_k = D^{\propto}_{T(t_{k+1})}(x\,(t_k)\,, y\,(t_k)\,; \alpha, \beta)$ for $k = 0, \dots, m-1$.

We can conclude this subsection by stressing that the non-regularity assumption has not prevented us from constructing a recurrent process. However, it is not one parameter but several parameters which reflect the technological progress between successive time periods.

4 Dynamical path in continuous time

Let us assume that the time interval between two time periods is infinitively small. As we shall see, this alternative time specification does not alter too much the expression of the proportional Luenberger indicator, its properties and the dynamical path.

4.1 Definition and notations

Definition 4.1.1 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t\geq t_0}$ be a dynamical path of $\{T(t)\}_{t\geq t_0}$. The dynamical path of $\{(x(t), y(t))\}_{t\geq t_0}$ is optimal and regular if and only if for all $t \geq t_0$ $(x(t), y(t)) \in \partial^{\propto}_{\alpha,\beta}(T(t))$ and there exists a continuous map $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\forall \Delta t > 0$*

(1) *the Proportional Distance Function satisfies the relation*

$$D^{\propto}_{T(t+\Delta t)}(x(t), y(t); \alpha, \beta) = H(\Delta t);$$

(2) *the function H is right-differentiable at 0, i.e.*

$$\exists \theta \in \mathbb{R}_+ : \lim_{\Delta t \to 0_+} \frac{H(\Delta t)}{\Delta t} = \theta$$

where Δt is the time interval.

To simplify the notation, we denote $d(t, T(t+\Delta t))$ the expression of the proportional distance $D^{\propto}_{T(t+\Delta t)}(x(t), y(t); \alpha, \beta)$.

The optimal path described in the above definition is said to be θ-regular path. We assume an instantaneous adaption of the production unit integrating technological progress. It is optimal as it characterizes the evolution of the technical efficiency frontier over time. The relation expressed in this definition is similar to that expressed in the discrete case. The main difference between both definitions rests on the instantaneous feature which prevents the production unit from admitting technological decay in the continuous case.

Lemma 4.1.2 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t\geq t_0}$ be an optimal and θ-regular dynamical path. If $\{(x(t), y(t))\}_{t\geq t_0}$ is θ-regular, we have*

$$x(t) = e^{-\theta t A} x_0 \quad \text{and} \quad y(t) = e^{\theta t B} y_0$$

where θ is the instantaneous variation rate of factors or products.

Proof: Let us assume that $(x(t+\Delta t), y(t+\Delta t))$ realizes the value $d(t, T(t+\Delta t))$ on $\partial^{\propto}_{\alpha,\beta}(T(t+\Delta t))$. For all $(i,j) \in I(x(t)) \times J(y(t))$, we have

$$x_i(t+\Delta t) = x_i(t) - x_i(t)\alpha_i d(t, T(t+\Delta t)) \tag{4.1}$$

and

$$y_j(t+\Delta t) = y_j(t) + y_j(t)\beta_j d(t, T(t+\Delta t)). \tag{4.2}$$

As $\Delta t > 0$, we can divide expressions 4.1 and 4.2 by Δt. We obtain

$$\frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} = -\frac{H(\Delta t)}{\Delta t}\alpha_i x_i(t)$$

and

$$\frac{y_j(t+\Delta t) - y_j(t)}{\Delta t} = \frac{H(\Delta t)}{\Delta t}\beta_j y_j(t).$$

As $H(0) = d(t, T(t)) = 0$ and the function H is right-differentiable at 0, we can evaluate the limit of the above both expressions when $\Delta t \to 0_+$

$$\lim_{\Delta t \to 0_+} \frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} = -\theta\alpha_i x_i(t)$$

and

$$\lim_{\Delta t \to 0_+} \frac{y_j(t+\Delta t) - y_j(t)}{\Delta t} = \theta\beta_j y_j(t).$$

Similarly, we can assume that $(x(t), y(t)$ realizes the value $d(t-\Delta t, T(t))$ on $\partial^{\propto}_{\alpha,\beta}(T(t))$. For all $(i,j) \in I(x(t)) \times J(y(t))$, we have:

$$x_i(t) = x_i(t-\Delta t) - x_i(t-\Delta t)\alpha_i d(t-\Delta t, T(t))$$

and

$$y_j(t) = y_j(t - \Delta t) + y_j(t - \Delta t)\beta_j d(t - \Delta t, T(t)).$$

Simple permutations yield:

$$\frac{x_i(t) - x_i(t - \Delta t)}{\Delta t} = -\frac{H(\Delta t)}{\Delta t}\alpha_i x_i(t - \Delta t)$$

and

$$\frac{y_j(t) - y_j(t - \Delta t)}{\Delta t} = \frac{H(\Delta t)}{\Delta t}\beta_j y_j(t - \Delta t).$$

As $d(t, T(t + \Delta t)) = H(\Delta t)$ and $\lim_{\Delta t \to 0_+} H(\Delta t) = 0$, we can evaluate the limits of both expressions. These are equal to

$$\lim_{\Delta t \to 0_+} \frac{x_i(t) - x_i(t - \Delta t)}{\Delta t} = -\theta\alpha_i x_i(t)$$

and

$$\lim_{\Delta t \to 0_+} \frac{y_j(t) - y_j(t - \Delta t)}{\Delta t} = \theta\beta_j y_j(t).$$

As the vectorial function $(x(t), y(t))$ is also left-differentiable at time period t, it is differentiable $\forall t \geq t_0$. As a result, for all $(i, j) \in I(x(t)) \times J(y(t))$,

$$\frac{dx_i(t)}{dt} = -\theta\alpha_i x_i(t) \qquad \text{and} \qquad \frac{dy_j(t)}{dt} = \theta\beta_j y_j(t).$$

We integrate both expressions from t_0 to t and we denote $x(t_0)$ by x_0 and $y(t_0)$ by y_0 to obtain

$$x(t) = e^{-\theta t A}x_0 \qquad \text{and} \qquad y(t) = e^{\theta t B}y_0. \square$$

The dynamical path described above is similar to that in the regular interval time in Section 3. However, it is 'smoother', *i.e.* the factors and the product are 'continuously' used in the same proportion.

As we have used the concept of proportional distance function, we are able to consider different weightings on the factors and the products. This allows us to have various and non constant proportions between input and output. For instance, in the case of a single product, the per capita output grows according to $y_t = e^{\theta t}F(e^{-\sigma t}x_0)$ where F is the production function, θ the growth rate of a product and σ the decline rate of factor at each time period. If the sum of the weightings are equal to 1, *i.e.* $\theta +\ \sigma$, we have a purely proportional indicator.

When the coefficient associated with a factor (product) is null, the growth (decline) rate is also null. This means that the decision unit does not have any room for manoeuvre for the inputs (outputs) considered. We can then refer back to a dynamics where only the product grows exponentially and where there is only one exponential decay with respect to the inputs.

Finally, the case in which the factor weightings vary according to time could also be considered. This is often what is reflected in the exponential specifications of technological progress. The growth rate of technological progress is then not regular.

4.2 The proportional Luenberger indicator

In continuous time, the proportional Luenberger indicator is equal to

$$\begin{aligned}
&L\big(x(t), y(t), x(t+\Delta t), y(t+\Delta t); \alpha, \beta\big) \\
&= \frac{1}{2\Delta_t}\Big[\Big(D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(t+\Delta t), y(t+\Delta t); \alpha, \beta\big)\Big) \\
&\qquad + \Big(D^{\propto}_{T(t+\Delta t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t+\Delta t)}\big(x(t+\Delta t), y(t+\Delta t); \alpha, \beta\big)\Big)\Big]
\end{aligned} \tag{4.3}$$

In the case of a Cobb-Douglas production function with constant returns to scale with $\alpha = \mathbb{1}^n$ and $\beta = 0$, $L\big(x(t), y(t), x(t+\Delta t), y(t+\Delta t); \alpha, \beta\big) = TECH_{t,t+\Delta t}$ where

$$TECH_{t,t+\Delta t} = \frac{1}{2}\Big[\frac{A(t+\Delta t) - A(t)}{A(t)} + \frac{A(t+\Delta t) - A(t)}{A(t+\Delta t)}\Big]. \tag{4.4}$$

If the path $\{x_t\}_{t\geq t_0}$ is optimal and A is assumed differentiable at time period t, we obtain

$$\lim_{\Delta t \to 0} L\big(x(t), y(t), x(t+\Delta t), y(t+\Delta t); \alpha, \beta\big) = \frac{dA(t)/dt}{A(t)}. \tag{4.5}$$

If the proportional distance function is continuously differentiable in time periods s and t, the proportional Luenberger indicator in continuous time is then defined by:

$$\mathcal{L}\big(x(t), y(t); \alpha, \beta\big) = -\left.\frac{\partial D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)}{\partial s}\right|_{s=t} \tag{4.6}$$

Lemma 4.2.1 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t\geq t_0}$ be a dynamical path. If $D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)$ is continuously differentiable in time periods s and t, then*

$$\mathcal{L}\big(x(t), y(t); \alpha, \beta\big) = -\left.\frac{dD^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)}{ds}\right|_{s=t} + \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big)}{\partial s}\right|_{s=t}.$$

Proof: Let $\Phi : [t_0, +\infty[\times [t_0, +\infty[$ be the map defined by

$$\Phi(s, t) = D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big).$$

It follows that $D^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big) = \Phi(\xi(s))$ where $\xi(s) = (s, s)$. Differentiating in s yields

$$\left.\frac{d\Phi(s, s)}{ds}\right|_{s=t} = \left.\frac{\partial \Phi(s, t)}{\partial s}\right|_{s=t} + \left.\frac{\partial \Phi(s, t)}{\partial t}\right|_{s=t}.$$

From which we obtain

$$\left.\frac{dD^{\propto}_{T(s)}\big(x(s),y(s);\alpha,\beta\big)}{ds}\right|_{s=t} = \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t),y(t);\alpha,\beta\big)}{\partial s}\right|_{s=t} + \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t),y(t);\alpha,\beta\big)}{\partial t}\right|_{s=t}.$$

Since $\left.\frac{\partial D^{\propto}_{T(s)}\big(x(t),y(t);\alpha,\beta\big)}{\partial t}\right|_{s=t} = \left.\frac{\partial D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)}{\partial s}\right|_{s=t}$, we deduce the result.□

From this Lemma, we derive the following corollary defining the proportional Luenberger indicator when the time interval is infinitively small.

Corollary 4.2.2 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t),y(t))\}_{t\geq t_0}$ be a dynamical path. If $D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)$ is continuously differentiable in time periods s and t, then*

$$\lim_{\Delta t\to 0} L\big(x(t),y(t),x(t+\Delta t),y(t+\Delta t);\alpha,\beta\big) = \mathcal{L}\big(x(t),y(t);\alpha,\beta\big).$$

Proof: By definition, we have

$$\begin{aligned}
&L\big(x(t),y(t),x(t+\Delta t),y(t+\Delta t);\alpha,\beta\big)\\
&= \frac{1}{2}\Big[\Big(\frac{D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t)}\big(x(t+\Delta t),y(t+\Delta t);\alpha,\beta\big)}{\Delta t}\Big)\\
&\qquad + \Big(\frac{D^{\propto}_{T(t+\Delta t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t+\Delta t)}\big(x(t+\Delta t),y(t+\Delta t);\alpha,\beta\big)}{\Delta_t}\Big)\Big].
\end{aligned}$$

Since the proportional distance function is continuously differentiable, we deduce

$$\begin{aligned}
&\lim_{\Delta t\to 0}\frac{1}{2}\left[\frac{D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t)}\big(x(t+\Delta t),y(t+\Delta t);\alpha,\beta\big)}{\Delta t}\right]\\
&+\lim_{\Delta t\to 0}\frac{1}{2}\left[\frac{D^{\propto}_{T(t+\Delta t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t+\Delta t)}\big(x(t+\Delta t),y(t+\Delta t);\alpha,\beta\big)}{\Delta_t}\right]\\
&= -\left.\frac{\partial D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)}{\partial s}\right|_{s=t}\\
&= \mathcal{L}\big(x(t),y(t);\alpha,\beta\big).\square
\end{aligned}$$

It follows that the proportional efficiency change in continuous time denoted $CEFCH$ is equal to

$$CEFCH = -\left.\frac{dD^{\propto}_{T(s)}\big(x(s),y(s);\alpha,\beta\big)}{ds}\right|_{s=t} \tag{4.7}$$

while the proportional technological change in continuous time denoted $CTECH$ is

$$CTECH = \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t),y(t);\alpha,\beta\big)}{\partial s}\right|_{s=t} \tag{4.8}$$

Corollary 4.2.3 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t),y(t))\}_{t\geq t_0}$ be an optimal and θ-regular dynamical path.*

If $D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)$ is continuously differentiable in time periods s and t, then

$$\mathcal{L}\big(x_0e^{-\theta tA},y_0e^{\theta tB};\alpha,\beta\big)=\theta.$$

This corollary states that a unique parameter characterizes the technological progress.

5 An exponential approach

In macroeconomics, growth is evaluated from exponential functions which capture technological progress over time. For this purpose we develop the concept of exponential distance function and establish its properties below. As we shall see, the exponential distance function has the main advantage to facilitate the productivity analysis in a macrodynamical framework. Indeed, the dynamics is independent on the size of the production unit (*i.e.* a country in a macroeconomics context).

5.1 Definition and properties

Let us first introduce the linear map $\Phi^{\delta}_{\alpha,\beta}:\mathbb{R}^{n+p}_{+}\longrightarrow\mathbb{R}^{n+p}_{+}$ defined for all $\delta\in\mathbb{R}$ as

$$\Phi^{\delta}_{\alpha,\beta}(x,y)=\big(e^{-\delta A}x,e^{\delta B}y\big) \tag{5.1}$$

Definition 5.1.1 *The map $D^{\exp}_{T(t)}:\mathbb{R}^{n+p}_{+}\longrightarrow\mathbb{R}\cup\{-\infty,+\infty\}$ defined by*

$$D^{\exp}_{T(t)}(x,y;\alpha,\beta)=\sup\{\delta:\Phi^{\delta}_{\alpha,\beta}(x,y)\in T(t)\} \tag{5.2}$$

is called the exponential Distance Function.

This function satisfies the properties gathered in the following proposition.

Proposition 5.1.2 *For all $(\alpha,\beta)\in[0,1]^n\times[0,1]^p$ the exponential distance function satisfies the following properties:*

(1) *$(x,y)\in T(t)$ if and only if $D^{\exp}_{T(t)}(x,y;\alpha,\beta)\geq 0$;*

(2) *For all $(x,y),(u,v)\in T(t)\wedge(-u,v)\geq(-x,y)\Longrightarrow D^{\exp}_{T(t)}(x,y;\alpha,\beta)\leq D^{\exp}_{T(t)}(u,v;\alpha,\beta)$;*

(3) *If $\alpha=\mathbb{1}^n$ and $\beta=0$, then $D^{\exp}_{T(t)}(x,y;\mathbb{1}^n,0)=-\ln(E^i_t(x,y))$;*

(4) *If $\alpha=0$ and $\beta=\mathbb{1}^p$, then $D^{\exp}_{T(t)}(x,y;0,\mathbb{1}^p)=\ln(E^o_t(x,y))$;*

(5) *$D^{\exp}_{T(t)}\big(\Phi^{\theta}_{\alpha,\beta}(x,y);\alpha,\beta\big)=D^{\exp}_{T(t)}\big(x,y;\alpha,\beta\big)-\theta$.*

Proof:

(1) immediately comes from the fact that $\Phi^0_{\alpha,\beta}$ is the identity map.

(2) Let us denote $K = \mathbb{R}^n_- \times \mathbb{R}^p_+$. If $(-u, v) \geq (-x, y)$, then

$$\{\delta : (e^{-\delta A}x, e^{\delta A}y) \in ((u, v) + K)\} \subset \{\delta : (e^{-\delta A}x, e^{\delta A}y) \in ((x, y) + K)\}.$$

In addition, we also have

$$\{\delta : (e^{-\delta A}u, e^{\delta A}v) \in ((u, v) + K)\} \subset \{\delta : (e^{-\delta A}x, e^{\delta A}y) \in ((u, v) + K)\}.$$

It follows

$$\{\delta : (e^{-\delta A}u, e^{\delta A}v) \in ((u, v) + K)\} \subset \{\delta : (e^{-\delta A}x, e^{\delta A}y) \in ((x, y) + K)\}.$$

which proves (2).

(3) If $\alpha = \mathbb{1}^n$ and $\beta = 0$, then

$$D^{\exp}_{T(t)}(x, y; \mathbb{1}^n, 0) = \sup\{\delta : (e^{-\delta}x, y) \in T(t)\}.$$

We set $\lambda = e^{\delta}$ and obtain

$$D^{\exp}_{T(t)}(x, y; \mathbb{1}^n, 0) = \ln\big(\sup\{\lambda > 0 : (\lambda^{-1}x, y) \in T(t)\}\big).$$

It follows that setting $\mu = \lambda^{-1}$, we have:

$$D^{\exp}_{T(t)}(x, y; \mathbb{1}^n, 0) = -\ln\big(\inf\{\mu \geq 0 : (\mu x, y) \in T(t)\}\big) = -\ln(E^i_t(x, y)).$$

(4) The proof of (4) is similar to the proof of (3).

(5) We have:

$$\begin{aligned}
D^{\exp}_{T(t)}\big(\Phi^{\theta}_{\alpha,\beta}(x, y); \alpha, \beta\big) &= \sup\{\delta : \Phi^{\delta}_{\alpha,\beta}\Phi^{\theta}_{\alpha,\beta}(x, y) \in T(t)\} \\
&= \sup\{\delta : (e^{-(\delta+\theta)A}x, e^{(\delta+\theta)B}y) \in T(t)\} \\
&= \sup\{\delta' : (e^{-\delta' A}x, e^{\delta' B}y) \in T(t)\} - \theta \\
&= D^{\exp}_{T(t)}\big(x, y; \alpha, \beta\big) - \theta. \square
\end{aligned}$$

Property 1 states that the exponential distance function characterizes the technology. This distance function satisfies the traditional monotonicity axiom by Property 2. In what follows, we denote '$\geqslant$' the partial order relative to this measure. Properties 3 and 4 indicates that the Farrell input measure and the Farrell output measure are special cases of this distance function. Property 5 allows us to deduce that this distance function is *exponential translation homothetic.*

6 Exponential Luenberger indicator

To be able to define the exponential Luenberger indicator, we first need to explain the concept of regularity in this setting.

Lemma 6.0.3 *Let $\{T(t)\}_{t=0,1,\ldots,t}$ be a family of embedded production sets satisfying Axioms T1-T4. Let $\{(x(\tau), y(\tau))\}_{\tau=0,1,\ldots,t}$ be an optimal and θ-regular dynamical path. For all $t < s$, we have*

$$D^{\exp}_{T(s)}(x(t), y(t); \alpha, \beta) = \theta(s-t).$$

Proof: If $\{(x(t), y(t))\}_{t=0,1,\ldots,t}$ is an optimal and θ-regular dynamical path, then $(x(\tau), y(\tau)) = (e^{-\theta\tau A}x(0), e^{\theta\tau B}y(0))$. Therefore $(x(t), y(t)) = (e^{-\theta t A}x(0), e^{\theta t B}y(0))$ and $(x(s), y(s)) = (e^{-\theta s A}x(0), e^{\theta s B}y(0))$. Hence $(x(s), y(s)) = (e^{-\theta(s-t)A}x(t), e^{\theta(s-t)B}y(t))$. However, as by definition, $(x(s), y(s))$ is a frontier point of $T(s)$, it follows that

$$\begin{aligned} D^{\exp}_{T(s)}(x(s), y(s); \alpha, \beta) = 0 &= D^{\exp}_{T(s)}(e^{-\theta(s-t)A}x(s), e^{\theta(s-t)B}y(s); \alpha, \beta) \\ &= D^{\exp}_{T(s)}(x(t), y(t); \alpha, \beta) - \theta(s-t) \end{aligned} \tag{6.1}$$

which ends the proof.□

By analogy to the definition provided in (3.2), we can introduce the so-called exponential Luenberger indicator. If $s \neq t$, we have

$$\begin{aligned} &L^{\exp}_{t,s}\big(x(t), y(t), x(s), y(s); \alpha, \beta\big) \\ &\quad = \frac{1}{2(s-t)}\Big[\Big(D^{\exp}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\exp}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)\Big) \\ &\qquad + \Big(D^{\exp}_{T(s)}\big(x(t), y(t); \alpha, \beta\big) - D^{\exp}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)\Big)\Big] \end{aligned} \tag{6.2}$$

As in the previous sections, we denote this exponential Luenberger indicator $PCH^{\exp}_{t,s}$. The latter can be decomposed into two components: the exponential efficiency change, $EFCH^{\exp}_{t,s}$, and the exponential technological change, $TECH^{\exp}_{t,s}$. It can be expressed as

$$PCH^{\exp}_{t,s} = EFCH^{\exp}_{t,s} + TECH^{\exp}_{t,s} \tag{6.3}$$

where

$$EFCH^{\exp}_{t,s} = \frac{1}{(s-t)}\Big[D^{\exp}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\exp}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)\Big] \tag{6.4}$$

and

$$\begin{aligned} TECH^{\exp}_{t,s} = \frac{1}{2(s-t)}\Big[&\Big(D^{\exp}_{T(s)}\big(x(t), y(t); \alpha, \beta\big) - D^{\exp}_{T(t)}\big(x(t), y(t); \alpha, \beta\big)\Big) \\ &+ \Big(D^{\exp}_{T(s)}\big(x(s), y(s); \alpha, \beta\big) - D^{\exp}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)\Big)\Big] \end{aligned} \tag{6.5}$$

Corollary 6.0.4 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t\geq t_0}$ be an optimal and θ-regular dynamical path. For all $t < s$, we have*

$$L^{\exp}_{t,s}\big(x(t), y(t), x(s), y(s); \alpha, \beta\big) = TECH^{\exp}_{t,s} = \theta.$$

Proof: Since $\{(x(t), y(t))\}_{t\geq t_0}$ is an optimal and θ-regular dynamical path, we have

$$D^{\exp}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) = 0 \quad \text{and} \quad D^{\exp}_{T(s)}\big(x(s), y(s); \alpha, \beta\big) = 0.$$

Hence

$$EFCH^{\exp}_{t,s} = 0$$

and

$$TECH^{\exp}_{t,s} = \frac{1}{2(s-t)}\Big[\Big(D^{\exp}_{T(s)}\big(x(t), y(t); \alpha, \beta\big) - D^{\exp}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)\Big)\Big].$$

From Lemma 6.0.3, it follows

$$TECH^{\exp}_{t,s} = \frac{1}{2(s-t)}\big[\theta(s-t) - \theta(t-s)\big] = \theta. \square$$

6.1 Examples

6.1.1 Cobb-Douglas parametric case

We consider the Cobb-Douglas technology defined in (2.11) and we assume that firms are efficient at each time period. If we set $\alpha = \mathbb{1}^n$ and $\beta = 0$, we have

$$D^{\exp}_{T(t)}(x(t), y(t); \mathbb{1}^n, 0) = D^{\exp}_{T(s)}(x(s), y(s); \mathbb{1}^n, 0) = EFCH^{\exp}_{t,s} = 0 \tag{6.6}$$

and

$$D^{\exp}_{T(t)}(x(s), y(s); \mathbb{1}^n, 0) = \sup\Big\{\delta : y(s) \leq A(t)\prod_{i=1}^{n}\big(e^{-\delta}x_i(t)\big)^{\gamma_i}\Big\} \tag{6.7}$$

with $\gamma_i > 0$ and $\sum_{i=1}^{n}\gamma_i = 1$. Therefore we have $PCH^{\exp}_{t,s} = TECH^{\exp}_{t,s}$. Taking the Log-transformation to the inequation in (6.7) yields

$$\begin{aligned}
D^{\exp}_{T(t)}(x(s), y(s); \mathbb{1}^n, 0) &= \sup\Big\{\delta : \ln(y(s)) \leq -\delta + \ln\big(A(t)\prod_{i=1}^{n}\big(x_i(t)\big)^{\gamma_i}\big)\Big\} \\
&= \sup\Big\{\delta : \delta \leq \ln\big(A(t)\prod_{i=1}^{n}\big(x_i(t)\big)^{\gamma_i}\big) - \ln(y(s))\Big\} \\
&= \sup\Big\{\delta : \delta \leq \ln\big(\frac{A(t)}{A(s)}\big)\Big\} = \ln(A(t)) - \ln(A(s))
\end{aligned} \tag{6.8}$$

Similarly,

$$D^{\exp}_{T(s)}(x(t), y(t); \mathbb{1}^n, 0) = \ln(A(s)) - \ln(A(t)),$$

which yields

$$TECH^{\exp}_{t,s} = \frac{\ln(A(s)) - \ln(A(t))}{(s-t)} \tag{6.9}$$

As A is continuously differentiable with respect to the time period, we obtain

$$\lim_{s \longrightarrow t} TECH^{\exp}_{t,s} = \frac{\ln(A(s)) - \ln(A(t))}{(s-t)} = \frac{d(\ln(A(t)))}{dt} = \frac{d(A(t))/dt}{A(t)}$$

which is equivalent to the Solow's formula of technological progress as we saw in Section 2.

6.1.2 Cobb-Douglas non-parametric case

At each time period we consider J observed firms. Let $\{(x_j(\tau), y_j(\tau))\}_{\tau=0,\dots,t}$ be the jth dynamical path. We assume that for all j and all τ we have $(x_j(\tau), y_j(\tau)) > 0$. As in Banker and Maindiratta (1986), we consider a class of Cobb-Douglas non parametric technologies defined by

$$T_{(\tau)} = \Big\{(x(\tau), y(\tau)) : x(\tau) \geq \prod_{j=1}^{J} x_j(\tau)^{\lambda_j};\ y(\tau) \leq \prod_{j=1}^{J} y_j(\tau)^{\lambda_j}; \sum_{j=1}^{J} \lambda_j = 1;\ \lambda \geq 0\Big\}. \tag{6.10}$$

The evaluation of $D^{\exp}_{T(t)}(x(s), y(s); \alpha, \beta)$ implies solving the following maximization program:

$$\begin{aligned} D^{\exp}_{T(t)}(x(s), y(s); \alpha, \beta) &= \max \delta \\ \text{subject to} \qquad & e^{-\delta A} x(s) \geq \prod_{j=1}^{J} x_j(t)^{\lambda_j} \\ & e^{\delta B} y(s) \leq \prod_{j=1}^{J} y_j(t)^{\lambda_j} \\ & \sum_j \lambda_j = 1, \lambda_j \geq 0 \end{aligned} \tag{6.11}$$

We can apply a Log-transformation to the above program to obtain:

$$
\begin{aligned}
D^{\exp}_{T(t)}(x(s), y(s); \alpha, \beta) &= \max \delta \\
\text{subject to} \qquad & \ln x(s) - \delta\alpha \geq \sum_{j=1}^{J} \lambda_j \ln x_j(t) \\
& \ln y(s) + \delta\beta \leq \sum_{j=1}^{J} \lambda_j \ln y_j(t) \\
& \sum_j \lambda_j = 1, \lambda_j \geq 0
\end{aligned}
\tag{6.12}
$$

where we denote $\ln z = (\ln z_1, ..., \ln z_{n+p})$ all vector z of $\mathbb{R}^{n+p}_+$ from which can be deduced $EFCH^{\exp}_{t,s}$, $TECH^{\exp}_{t,s}$ and $PCH^{\exp}_{t,s}$.

7 Conclusion

We have introduced Farrell technical efficiencies in simple Solow models by adopting a Luenberger-type approach. This introduction has allowed us to make an analogy between a firm's behavior in a microeconomic setting and a country's behavior in a macroeconomic setting, both in a discrete time framework and a continuous time framework. In both cases, we were able to estimate the total factor productivity without having specified the production function. We were also able to evaluate production inefficiencies and therefore identify possible aggregate gains.

References

[1] Allen, R.C., (1991), Entrepreneurship, Total Factor Productivity, and Economic Efficiency: Landes, Solow, and Farrell Thirty Years later, in P. Higonnet, D. S. Landes and H. Rosovsky (ed.) Favorites of Fortune: Technology, Growth, and Economic Development Since the Industrial Revolution, Harvard University Press, Cambridge, Mass, $203-220$.

[2] Banker, R.D., A. Charnes and W.W. Cooper, (1984), Some Models for Estimating Technical and Scale Efficiency in Data Envelopment Analysis, Management Science, 30, $1078-1092$.

[3] Banker, R.D. and A. Maindiratta, (1986), Piecewise Loglinear Estimation of Efficient Production Surfaces, Management Science, 32, 126-135.

[4] Boussemart, J.P., W. Briec, K. Kerstens and J.C. Poutineau, (2003), Luenberger and Malmquist Productivity Indices: Theoretical Comparisons and Empirical Illustration, Bulletin of Economic Research, 55(4), $391-405$.

[5] Briec, W., (1997), A Graph-Type Extension of Farrell Technical Efficiency Measure, Journal of Productivity Analysis, 8(1), $95 - 110$.

[6] Caves, D., L. Christensen and E. Diewert, (1982), The Economic Theory of Index Numbers, and the Measurement of Input-Output Productivity, Econometrica, 50, $1393 - 1414$.

[7] Chambers, R., Y. Chung and R. Färe, (1996), Benefit and Distance Functions, Journal of Economic Theory, 70, $407 - 419$.

[8] Chambers, R., Y. Chung and R. Färe, (1998), Profit, Directional Distance Functions, and Nerlovian Efficiency, Journal of Optimization Theory and Applications, 98, $351 - 364$.

[9] Chambers, R., R. Färe and B. Grosskopf, (1996), Productivity Growth in APEC Countries, Pacific Economic Review, 1, $181 - 190$.

[10] Charnes, A., W.W. Cooper and E. Rhodes, (1978), Measuring the Efficiency of Decision-Making Units, European Journal of Operational Research, 3, $429 - 444$.

[11] Del Gatto, M., A. Di Liberto and C. Petraglia, (2011), Measuring Productivity, Journal of Economic Surveys, 25(5), $952 - 1008$.

[12] Färe, R., S. Grosskopf, B. Lindgren and P. Roos, (1992), Productivity Changes in Swedish Pharmacies 1980-1989: A Non Parametric Malmquist Approach, Journal of Productivity Analysis, 3, $85 - 101$.

[13] Färe, R., S. Grosskopf and C.A.K. Lovell, (1985), The Measurement of Efficiency of Efficiency of Production, Kluwer Nijhof Publishing.

[14] Farrell, M.J., (1957), The Measurement of Productive Efficiency, Journal of the Royal Statistical Society, 120, $253 - 281$.

[15] Luenberger, D.G., (1992a), Benefit Function and Duality, Journal of Mathematical Economics, 21, $461 - 481$.

[16] Luenberger, D.G., (1992b), New Optimality Principles for Economic Efficiency and Equilibrium, Journal of Optimization Theory and Application, 75, $211 - 264$.

[17] Luenberger, D.G., (1995), Microeconomic Theory, Boston: McGraw Hill.

[18] ten Raa, T. and P. Mohnen, (2002), Neoclassical Growth Accounting and Frontier Analysis: a Synthesis, Journal of Productivity Analysis, 18(2), $111 - 128$.

[19] Russell, R.R., (1985), Measures of Technical Efficiency, Journal of Economic Theory, 35(1), 109-126.

[20] Russell, R.R., (1987), On the Axiomatic Approach to the Measurement of Technical Efficiency, in W. Eichhorn (ed.) Measurement in Economics: Theory and Application of Economic Indices, Physica-Verlag, 207-227.

[21] Russell, R.R. and W. Schworm (2011), Properties of inefficiency indexes on input-output space, Journal of Productivity Analysis, 36 (2), 143-156.

[22] Shephard, R.W., (1974), Indirect Production Functions, Meisenheim am Glan: Verlag Anton Ain.

[23] Solow, R., (1957), Technical Change and the Aggregate Production Function, Review of Economics and Statistics, 393, $312-320$.

[24] Tulkens, H. and P. Vanden Eeckaut, (1995), Non-Parametric Efficiency, Progress and Regress Measures for Panel Data: Methodological Aspects, European Journal of Operational Research, 80(3), $474-499$.

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